Nonlinear physics: Hamiltonian Chaos

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Outline

- **Hamiltonian systems (Example Hénon-Heiles system)**
  - ✓**Equations of motion**
  - ✓**Chaos**
  - ✓**Poincaré Surface of Section**
  - ✓**Variational equations**
  - ✓**Lyapunov exponents**

Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\underbrace{q_1,q_2,\ldots,q_N}_{\text{positions}},\ \underbrace{p_1,p_2,\ldots,p_N}_{\text{momenta}})$$

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The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0), q_2(0),\ldots,q_N(0), p_1(0), p_2(0),\ldots,p_N(0))$$

is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad , \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

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Phase space: the 2N dimensional space defined by variables $q_1,q_2,\ldots,q_N,\ p_1,p_2,\ldots,p_N$

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$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

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THE ASTRONOMICAL JOURNAL VOLUME 69, NUMBER 1 FEBRUARY 1964

The Applicability of the Third Integral Of Motion: Some Numerical Experiments

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Princeton University Observatory, Princeton, New Jersey

(Received 7 August 1963)

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Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

Chaos

Definition [Devaney (1989)]

Let V be a set and $f : V \to V$ a map on this set.

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3. **periodic points are dense in V.**

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There exist points arbitrarily close to **x** which eventually separate from **x** by at least δ under iterations of **f**.

Not all points near **x** need eventually move away from **x** under iteration, but there must be at least one such point in every neighborhood of **x**.

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Consequently, the dynamical system cannot be decomposed into two disjoint invariant open sets.

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3. **An element of regularity** because it has periodic points which are dense.

Usually, in physics and applied sciences, people use the notion of chaos in relation to the sensitive dependence on initial conditions.

Regular vs Chaotic orbits

Hénon-Heiles system

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

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For H=0.125 we get a regular and a chaotic orbit with initial conditions (ICs):

$x=0$, $y=0.1$, $p_y=0$ and **$x=0$, $y=-0.25$, $p_y=0$.**

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Orbit **Perturbed**

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		Orbit	**Perturbed**
t=	100	x= 0.132995718333307644	0.132995718337263064

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		Orbit	Perturbed
t= 100	x=	0.13299571833307644	0.13299571833727263064
t= 5000	x=	0.37699928389102310	0.376999283870156576

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t=	100	x= 0.13299571833307644	0.13299571833**7263064**
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t=	10000	x=-0.15909458335685522 4	-0.15909458334**1260309**

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t=	50000	x= 0.10199240073995576 0	0.10199240025396132 1

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t=	10000	x=-0.159094583356855224	-0.159094583341260309
t=	50000	x= 0.101992400739955760	0.101992400253961321
t=	100000	x=-0.381120533746511780	-0.381120533327258870

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t= 100	x=	0.13299571833307644	0.13299571833726306 4
t= 5000	x=	0.37699928389102310	0.37699928370156576
t= 10000	x=	-0.15909458335685522 4	-0.15909458334126030 9
t= 50000	x=	0.10199240073995576 0	0.10199240025396132 1
t=100000	x=	-0.38112053374651178 0	-0.38112053332725887 0
t= 100	x=	0.09027281773516783 5	0.09027282135576866 8

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t=	50000	x= 0.10199240073995576	0.10199240025396132
t=	100000	x=-0.38112053374651178	-0.38112053327258870
t=	100	x= 0.09027281773516783	0.09027282135576866
t=	200	x= 0.29503168748224928	0.29503188485862563

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t=100000	x=-0.38112053374651178	-0.38112053332725887
t= 100	x= 0.09027281773516783	0.09027282135576866
t= 200	x= 0.29503168748224928	0.29503188485862563
t= 300	x= 0.51522633010945018	0.51522544048069329

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t=	100000	x=-0.381120533746511780	-0.381120533327258870
t=	100	x= 0.090272817735167835	0.090272821355768668
t=	200	x= 0.295031687482249283	0.295031884858625637
t=	300	x= 0.515226330109450181	0.515225440480693297
t=	400	x= 0.063441889347425867	0.061359558551008345

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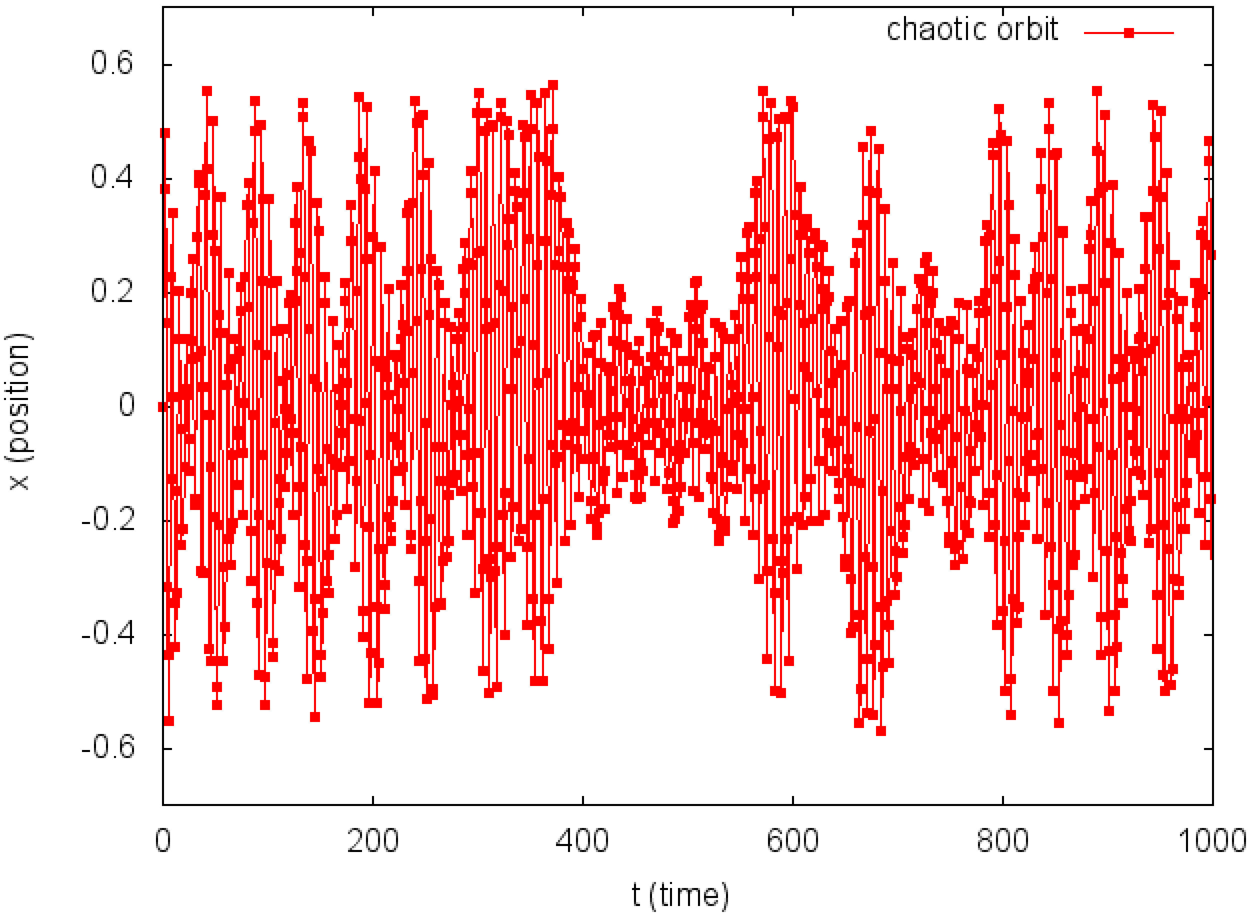
t= 100	x= 0.090272817735167835	0.090272821355768668
t= 200	x= 0.295031687482249283	0.295031884858625637
t= 300	x= 0.515226330109450181	0.515225440480693297
t= 400	x= 0.063441889347425867	0.061359558551008345
t= 500	x= 0.078357719290523528	-0.270811022674341095

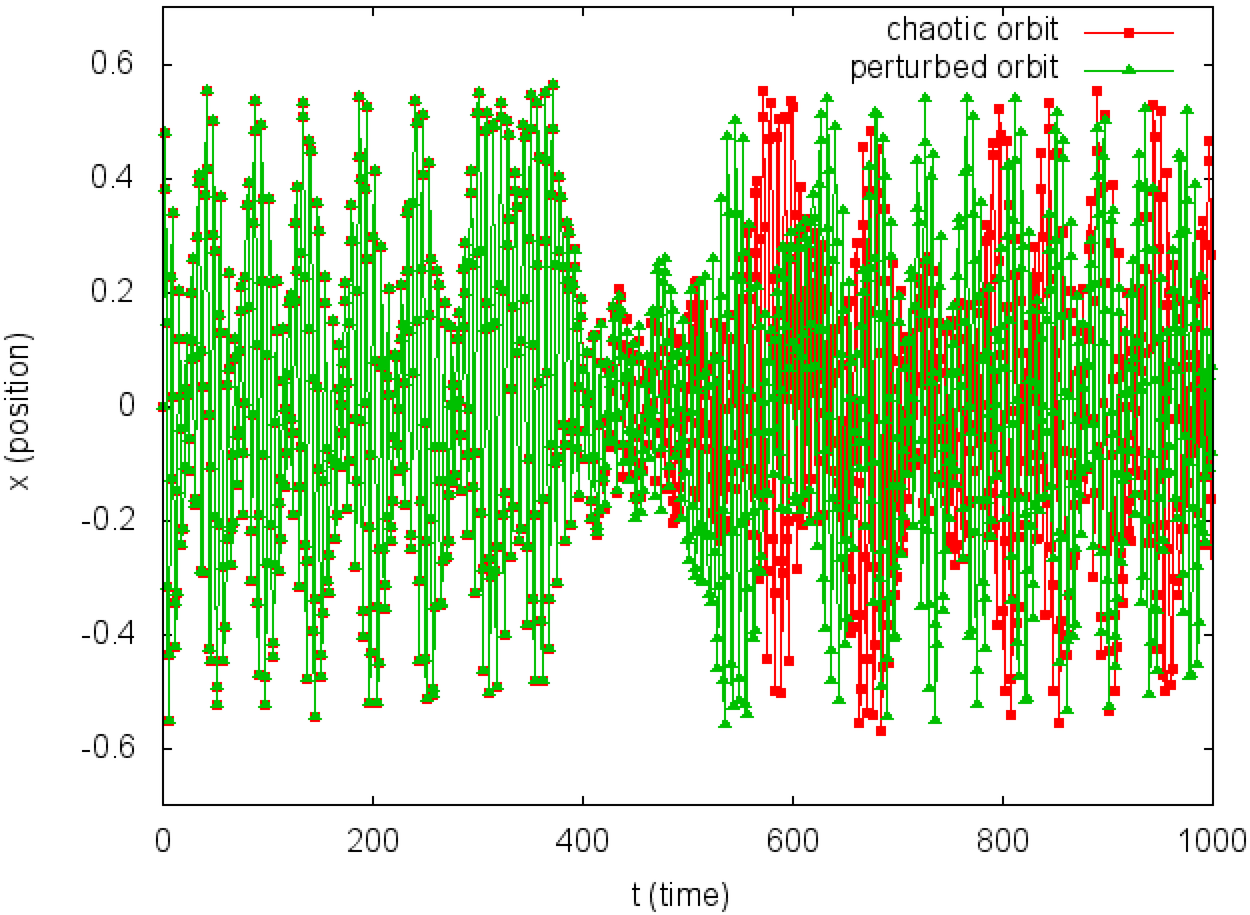
Regular vs Chaotic orbits

Chaotic orbit



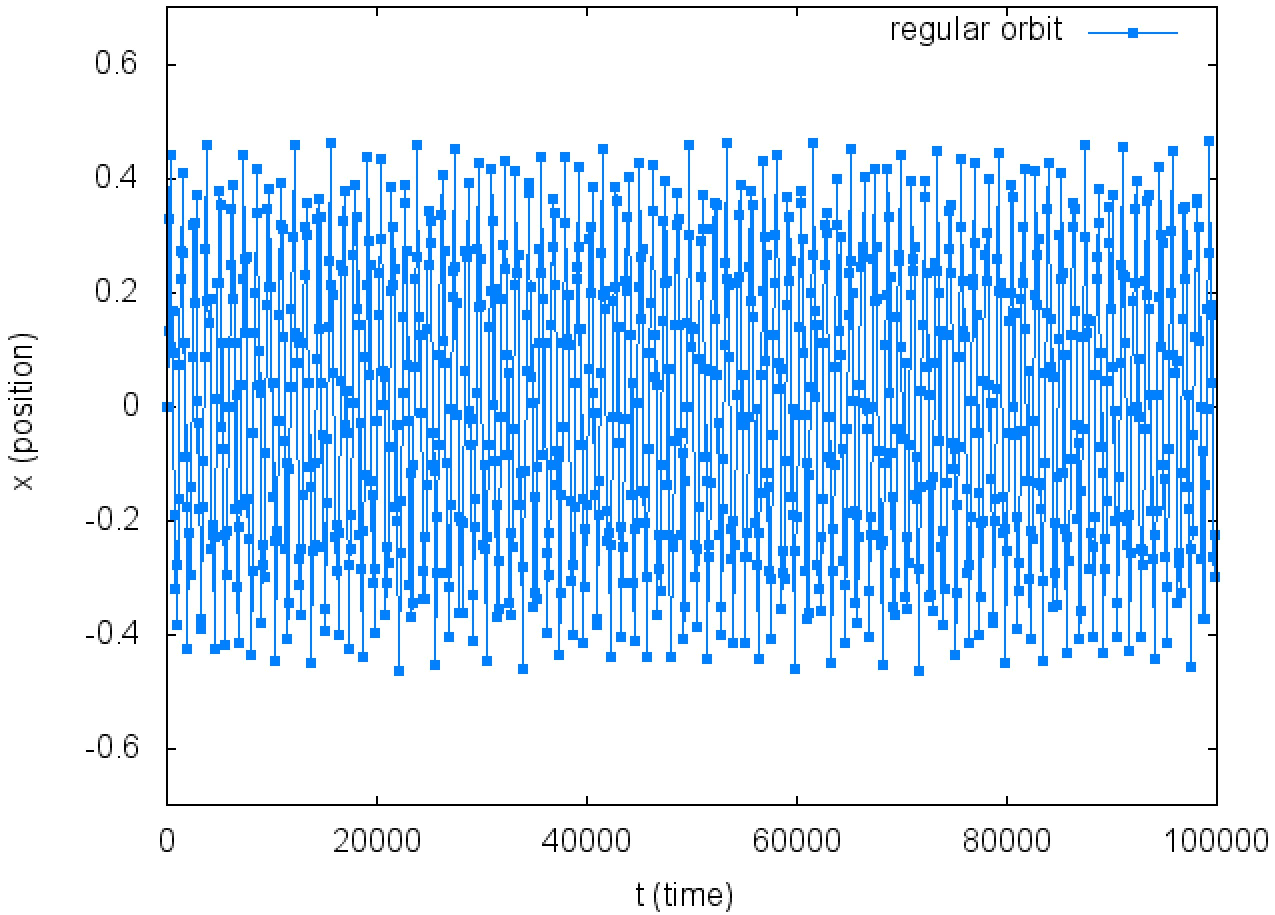
Regular vs Chaotic orbits

Chaotic orbit and its perturbation



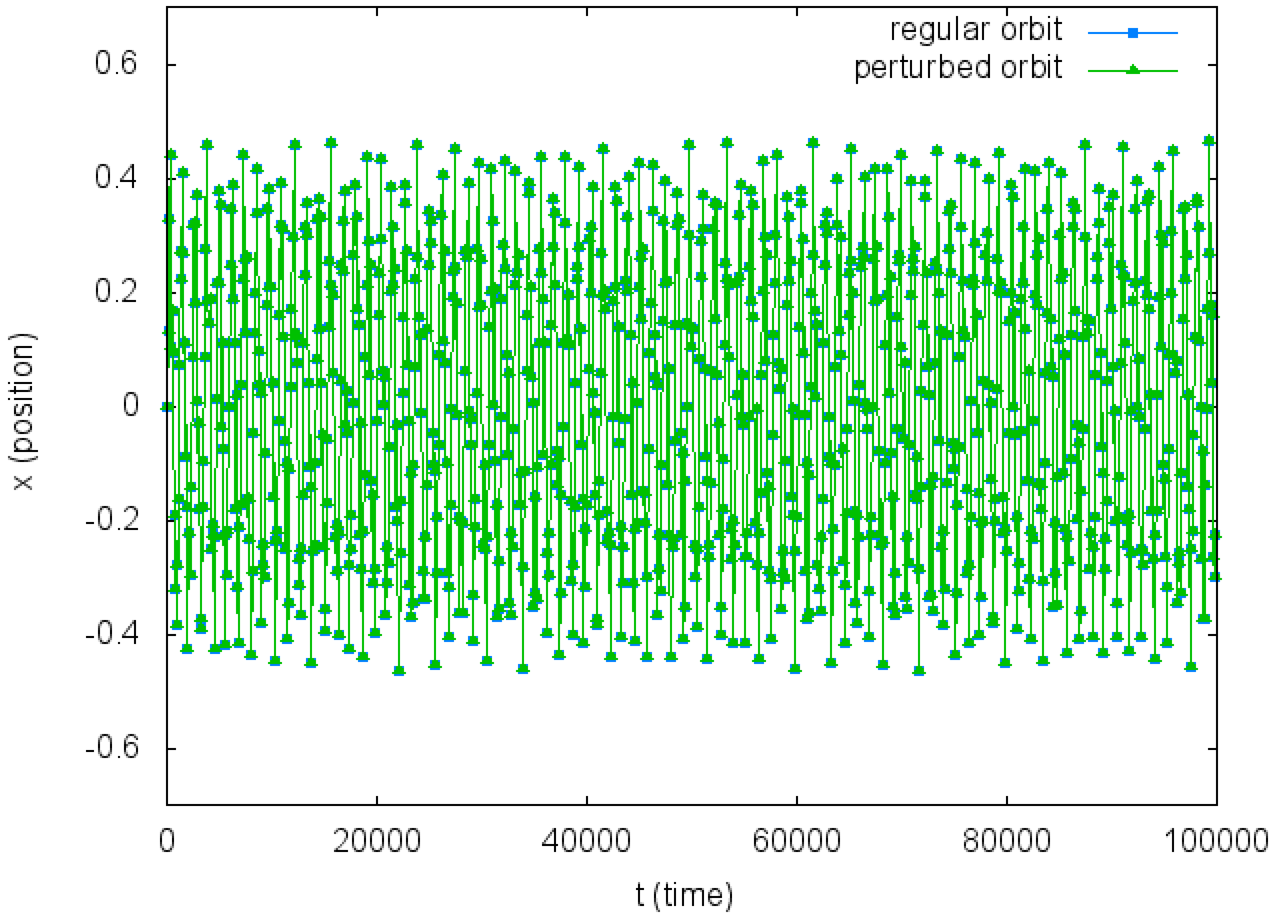
Regular vs Chaotic orbits

Regular orbit



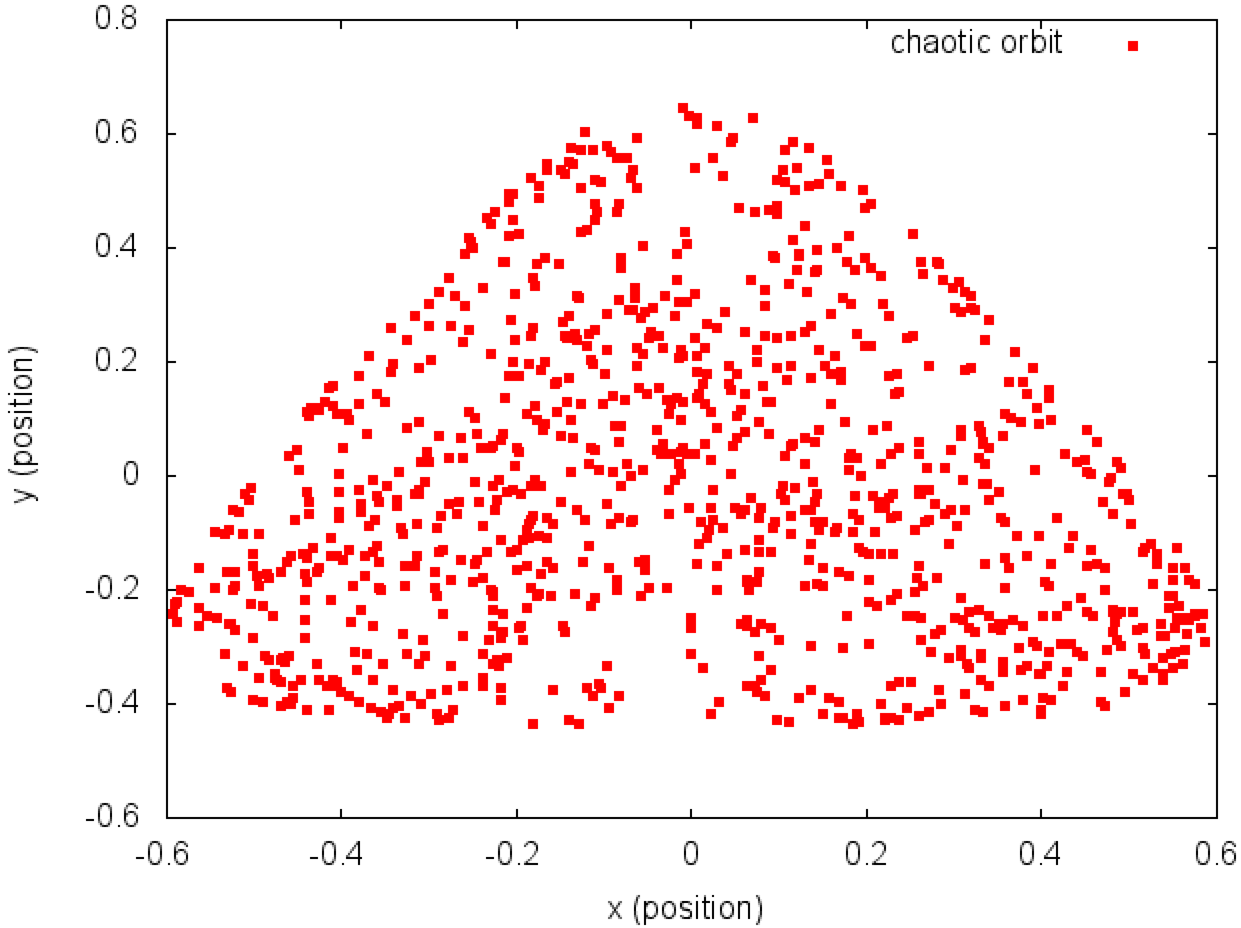
Regular vs Chaotic orbits

Regular orbit and its perturbation



Regular vs **Chaotic** orbits

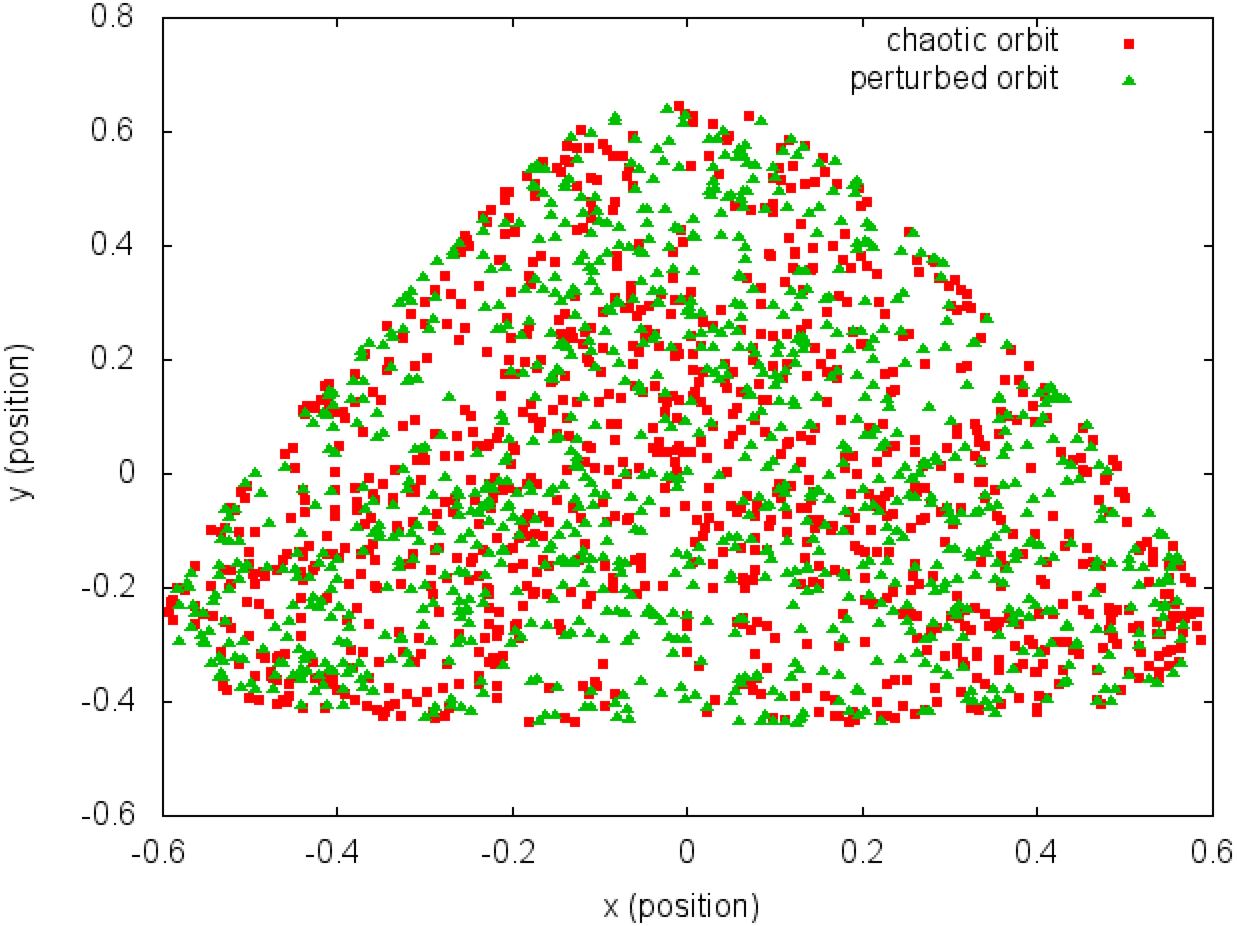
Chaotic orbit



Results for $0 \leq t \leq 10^5$

Regular vs Chaotic orbits

Chaotic orbit and its perturbation



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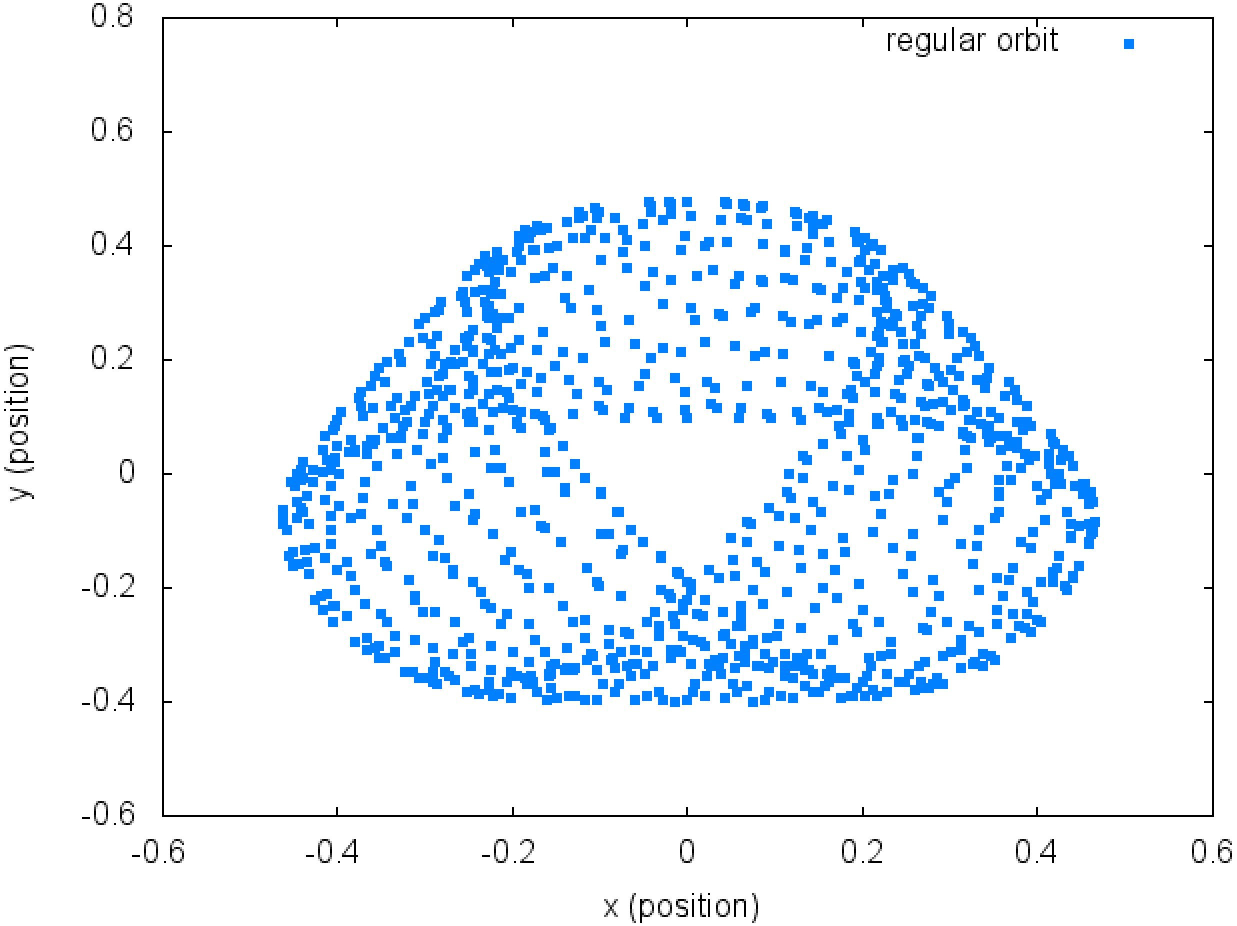
Regular orbit



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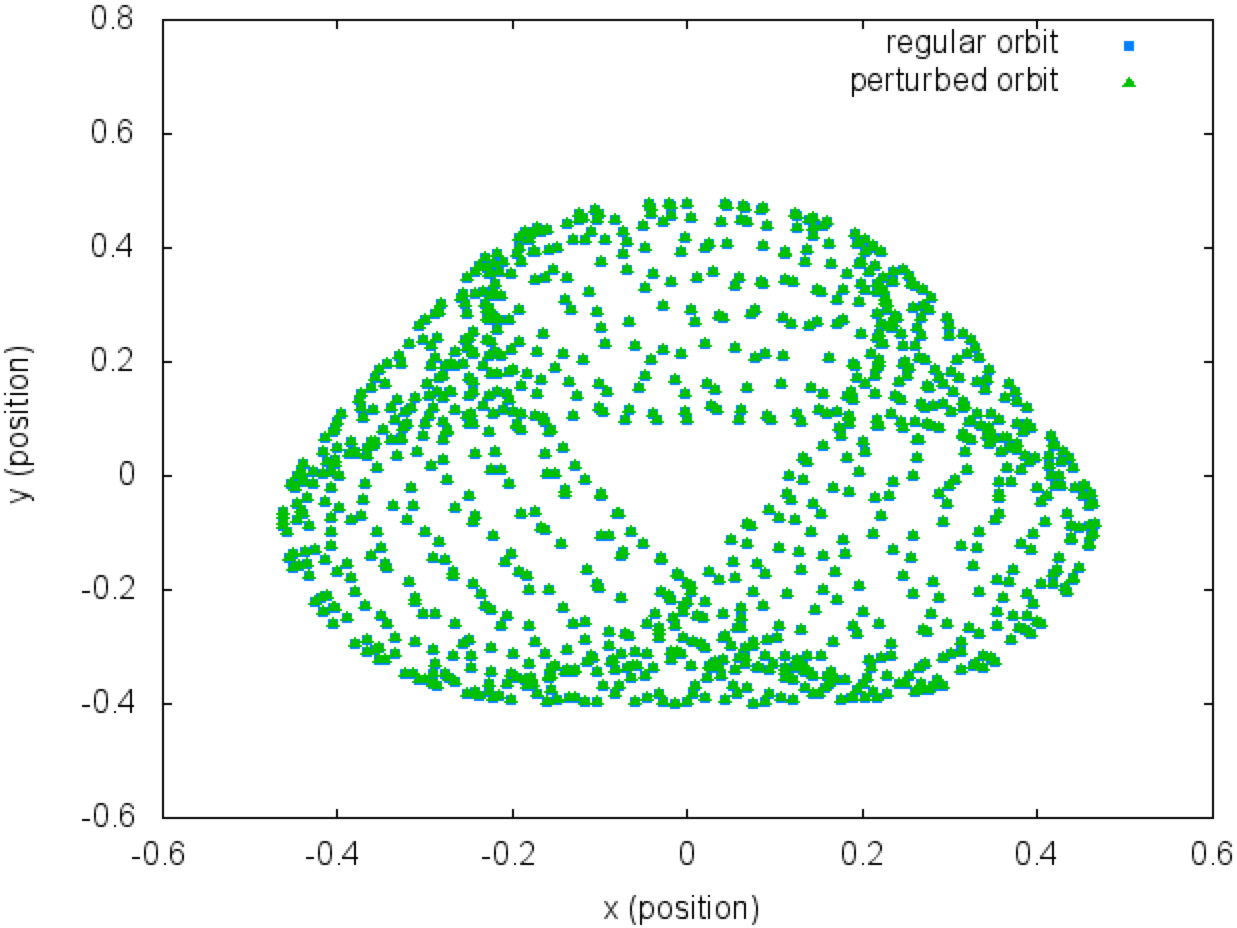

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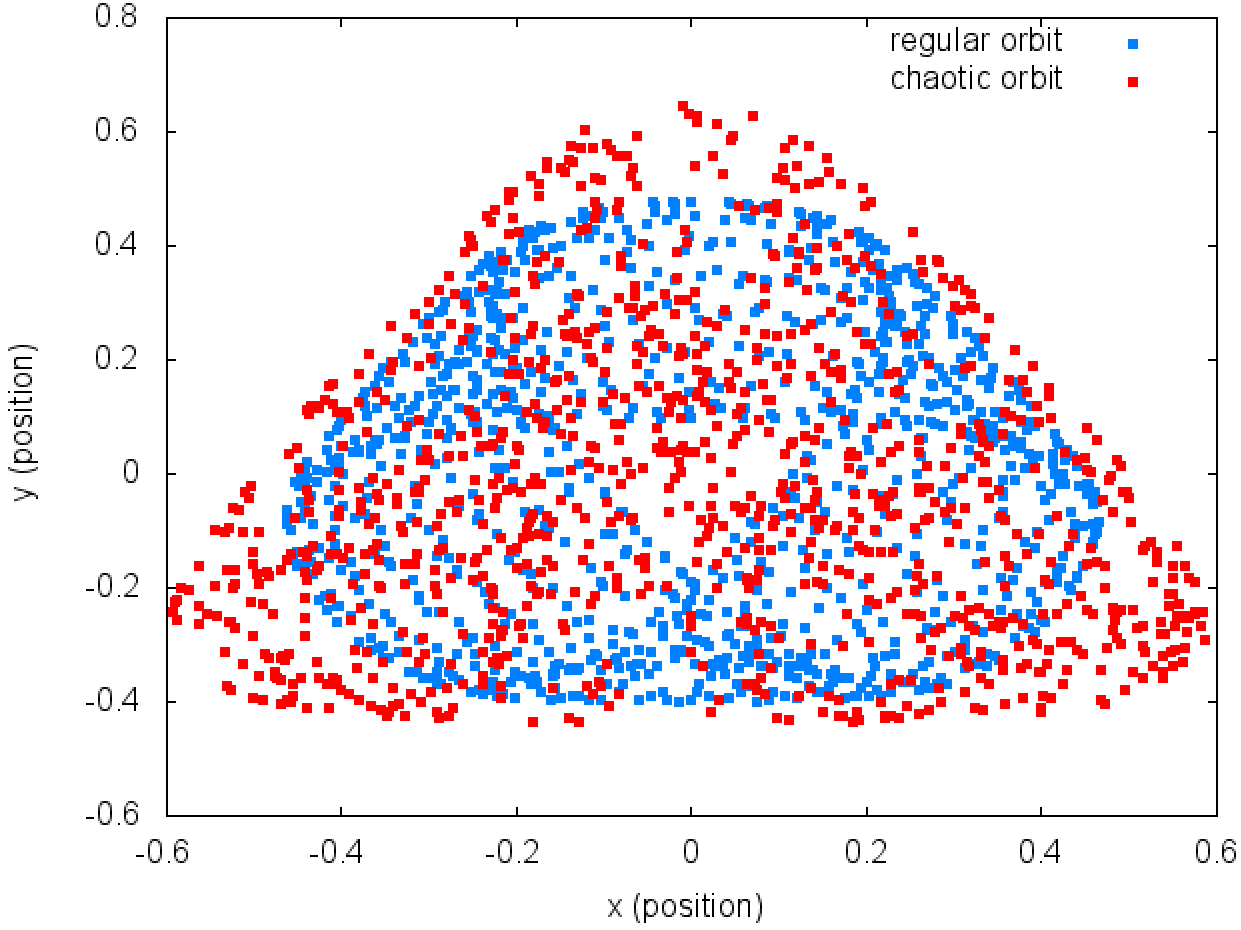
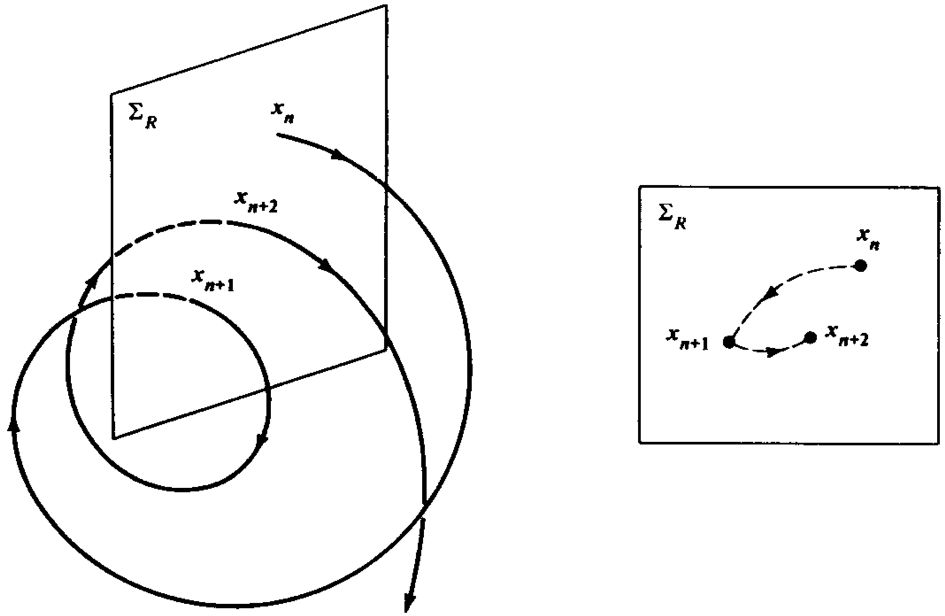


Results for $0 \leq t \leq 10^5$

Poincaré Surface of Section (PSS)

We can constrain the study of an N+1 degree of freedom Hamiltonian system to a 2N-dimensional subspace of the general phase space.



Lieberman & Lichtenberg, 1992, *Regular and Chaotic Dynamics*, Springer.

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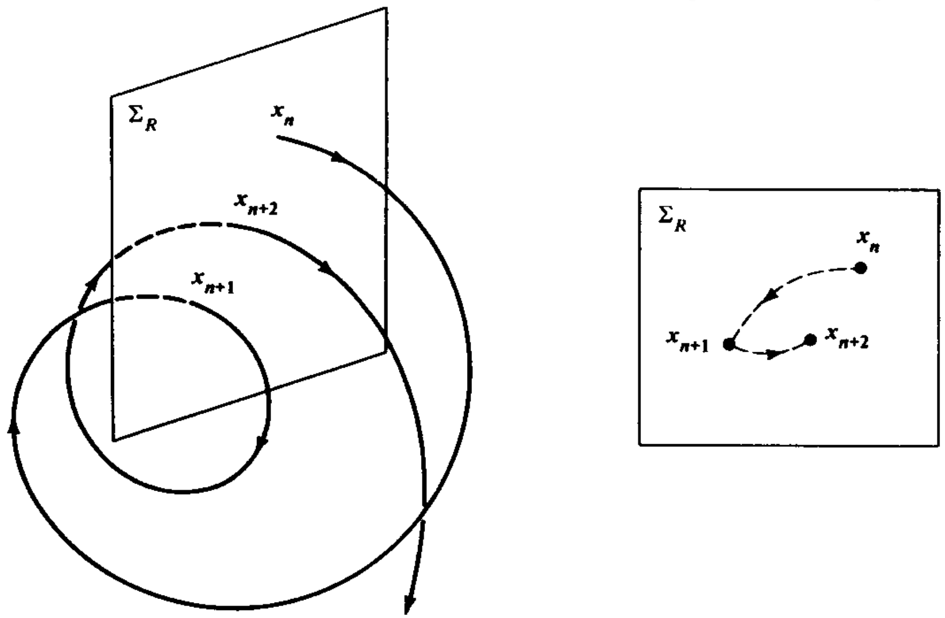
We can constrain the study of an N+1 degree of freedom Hamiltonian system to a 2N-dimensional subspace of the general phase space.



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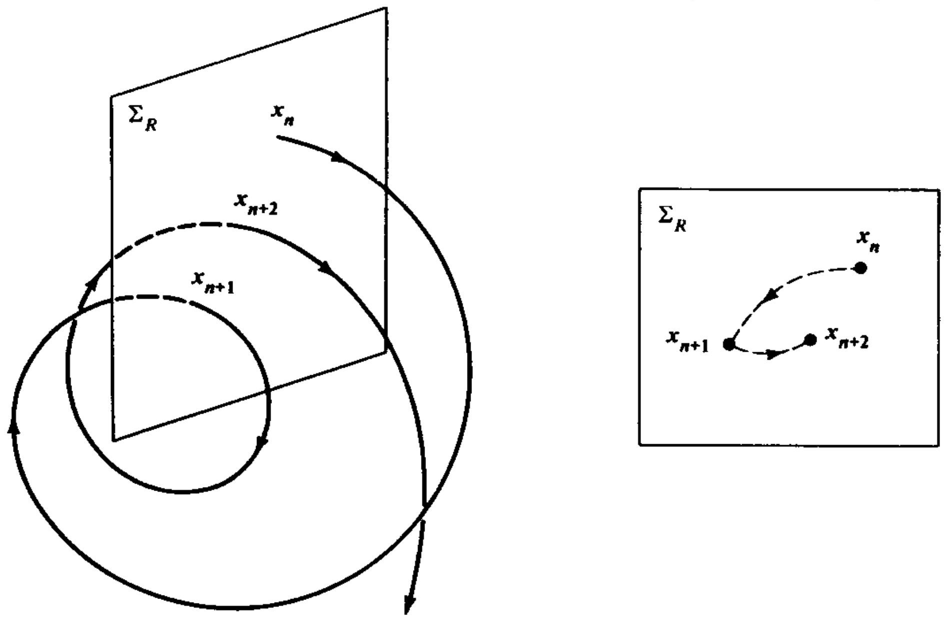

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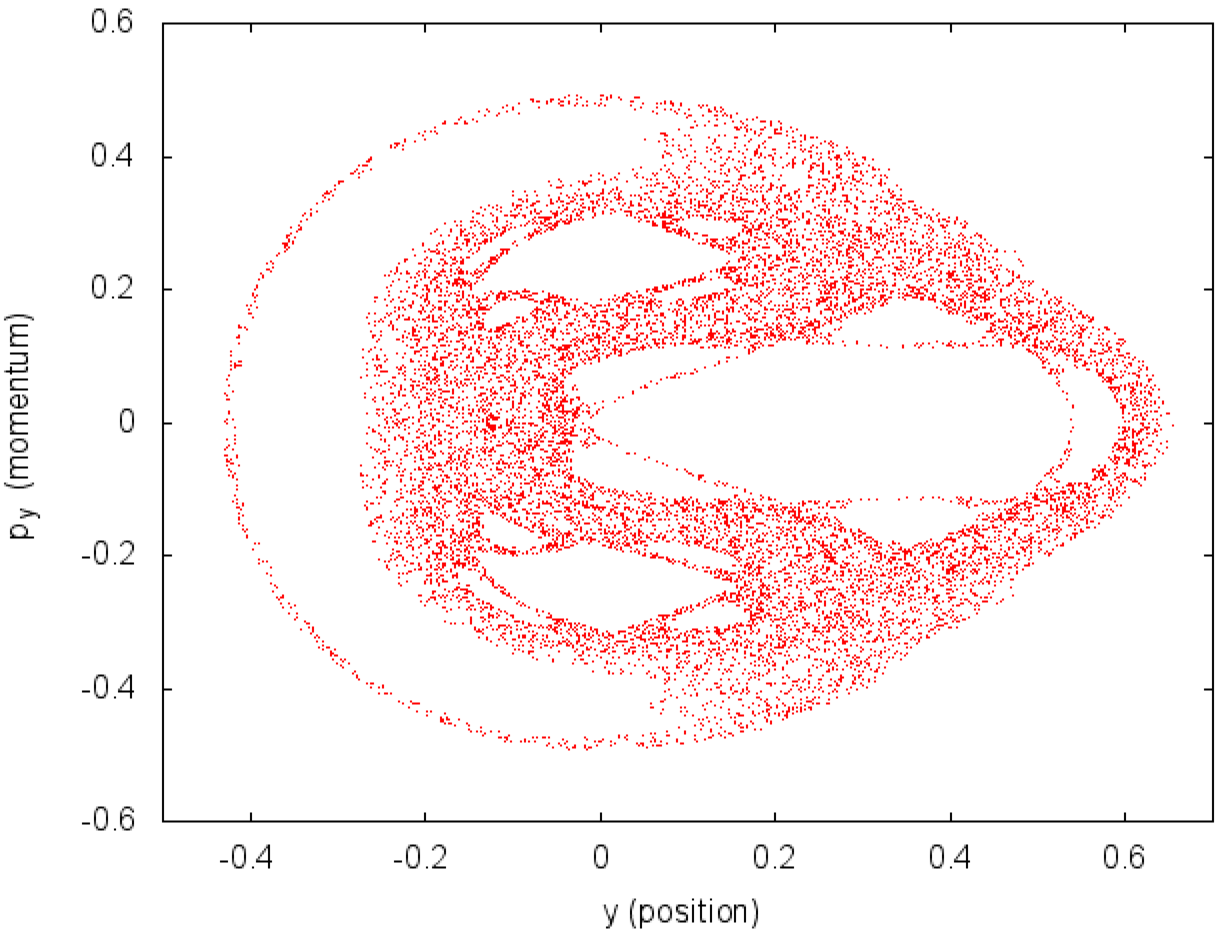
In this sense an N+1 degree of freedom Hamiltonian system corresponds to a 2N-dimensional map.

Hénon-Heiles system: PSS (x=0)



Chaotic orbit

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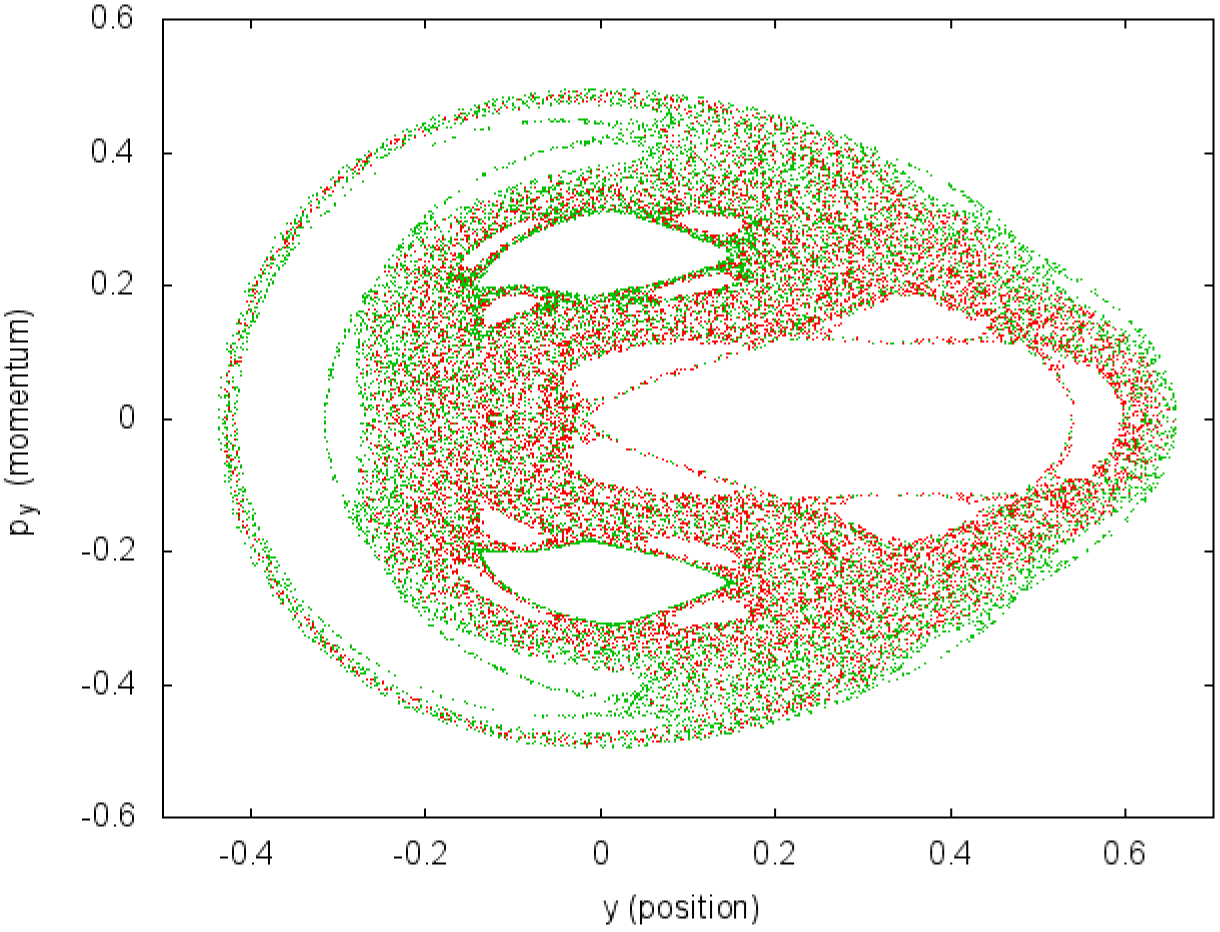

Chaotic orbit - Perturbed chaotic orbit

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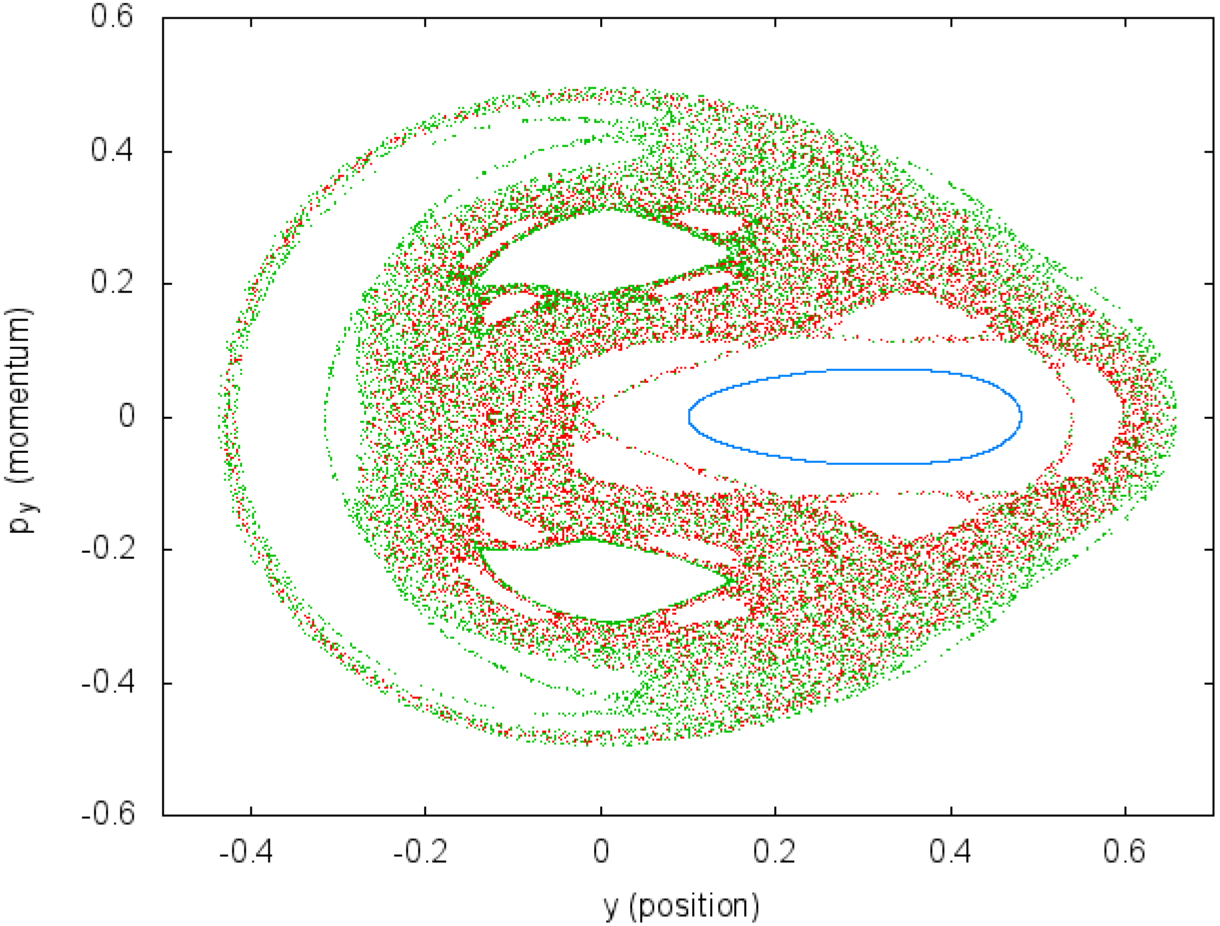

Chaotic orbit - Perturbed chaotic orbit
Regular orbit

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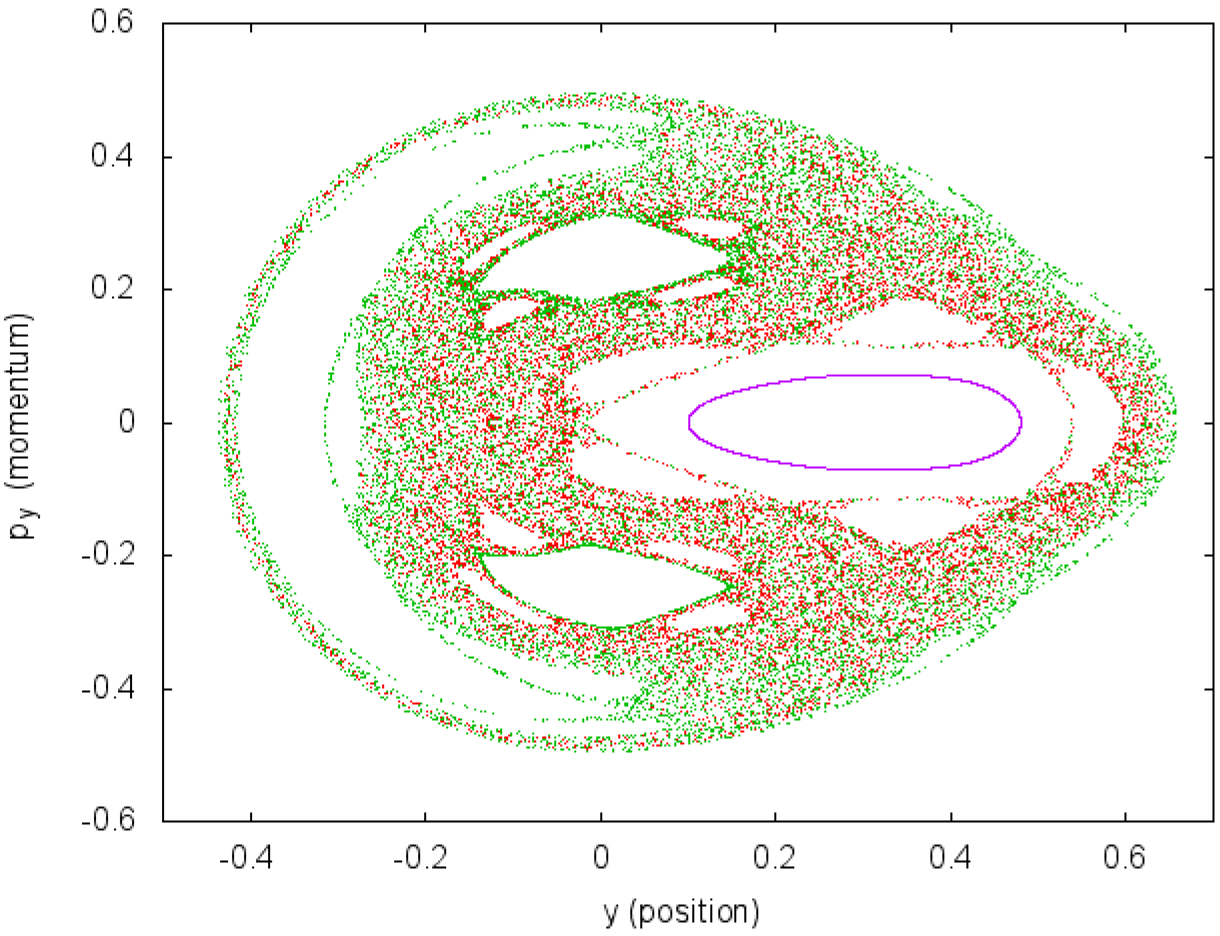

Chaotic orbit - **Perturbed chaotic orbit**

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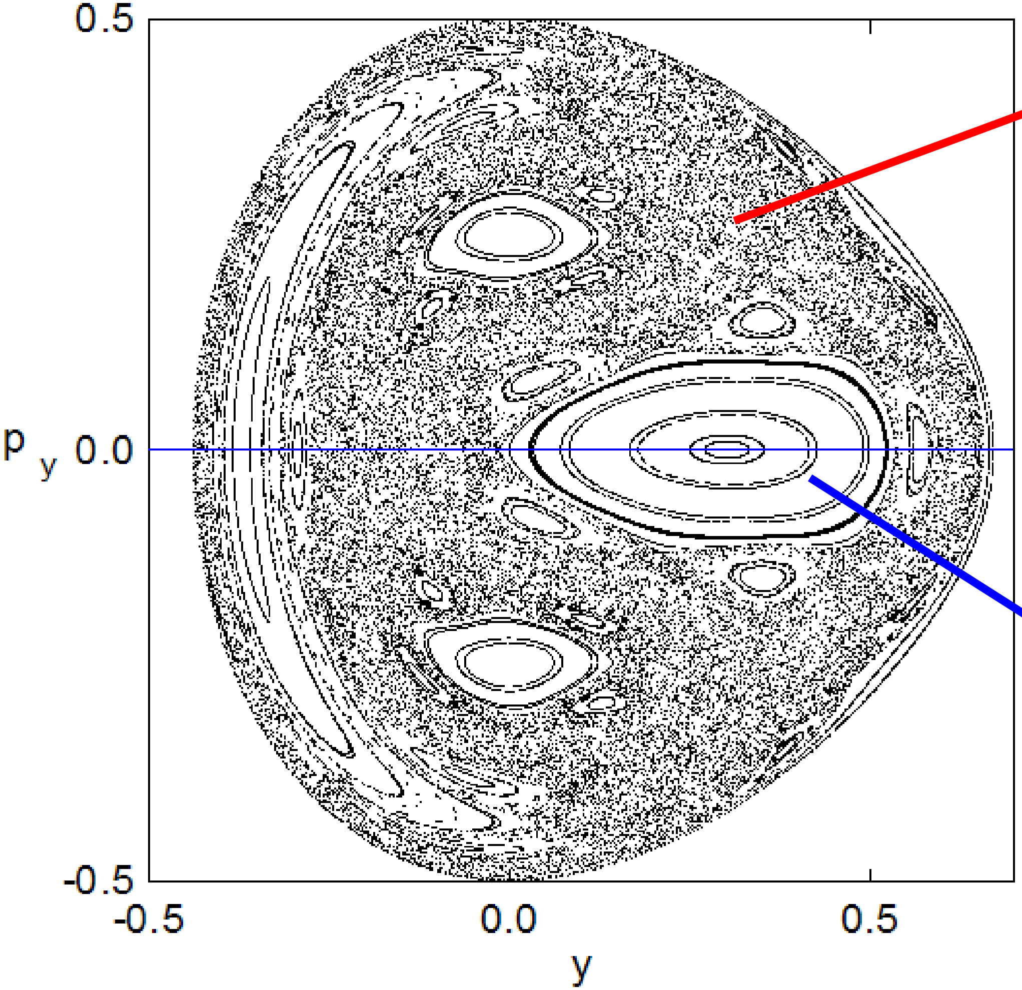
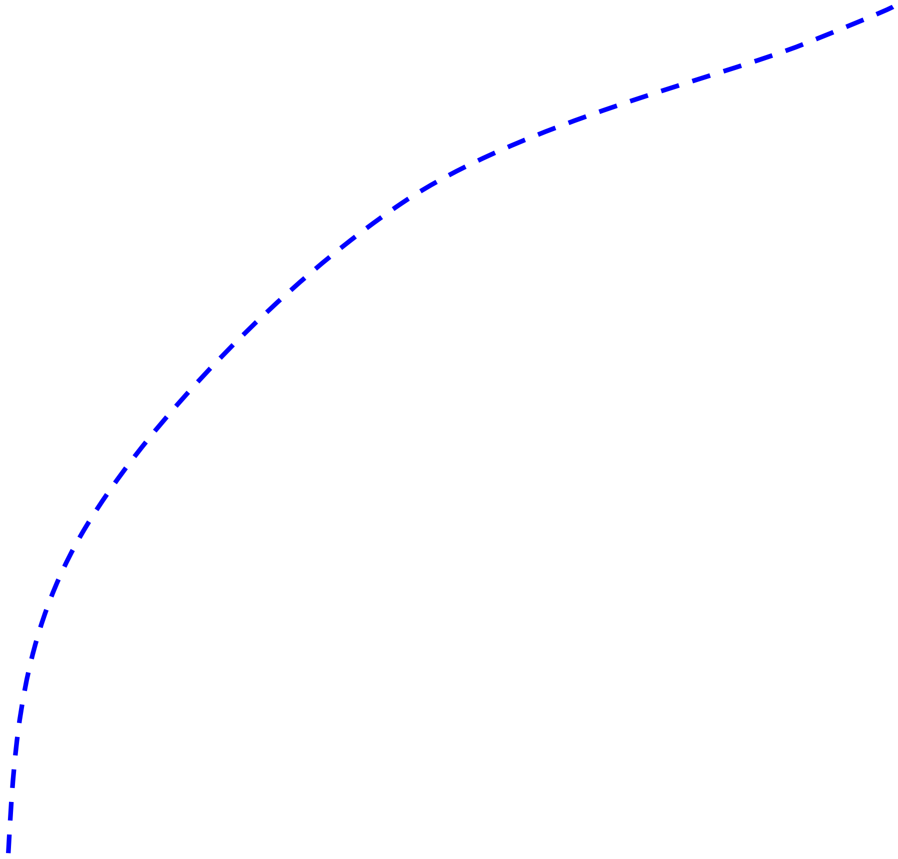


Chaotic motion
Chaotic sea

Regular motion
Island of stability

Computation of the PSS



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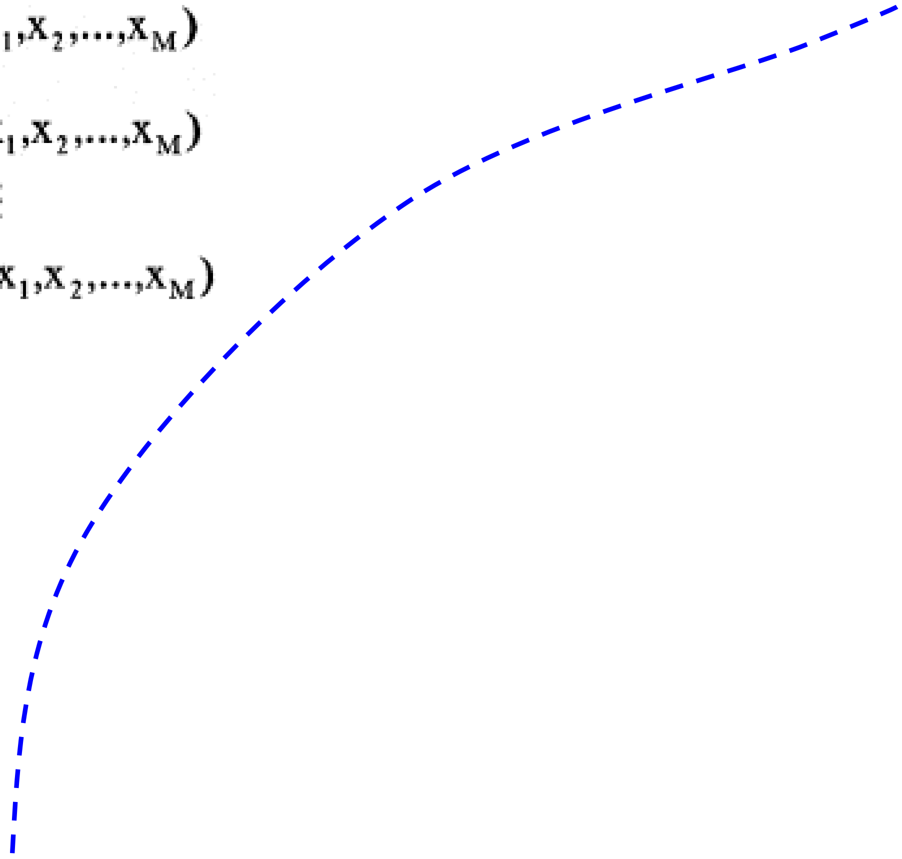

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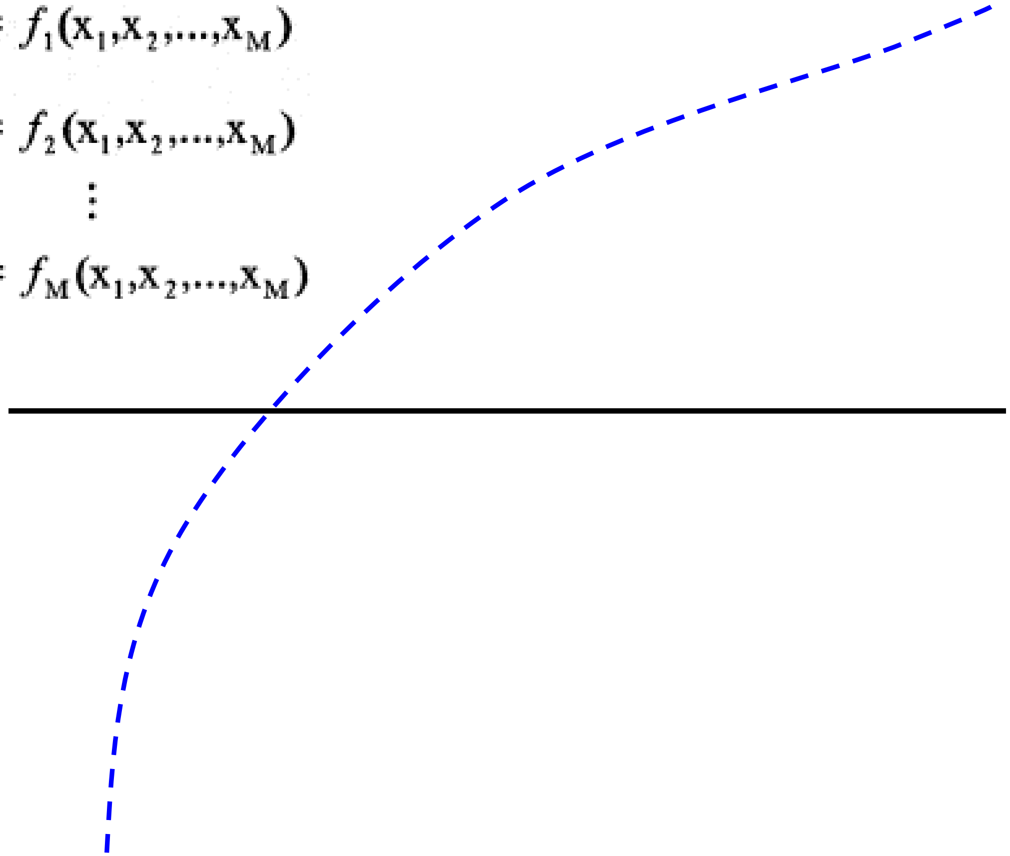

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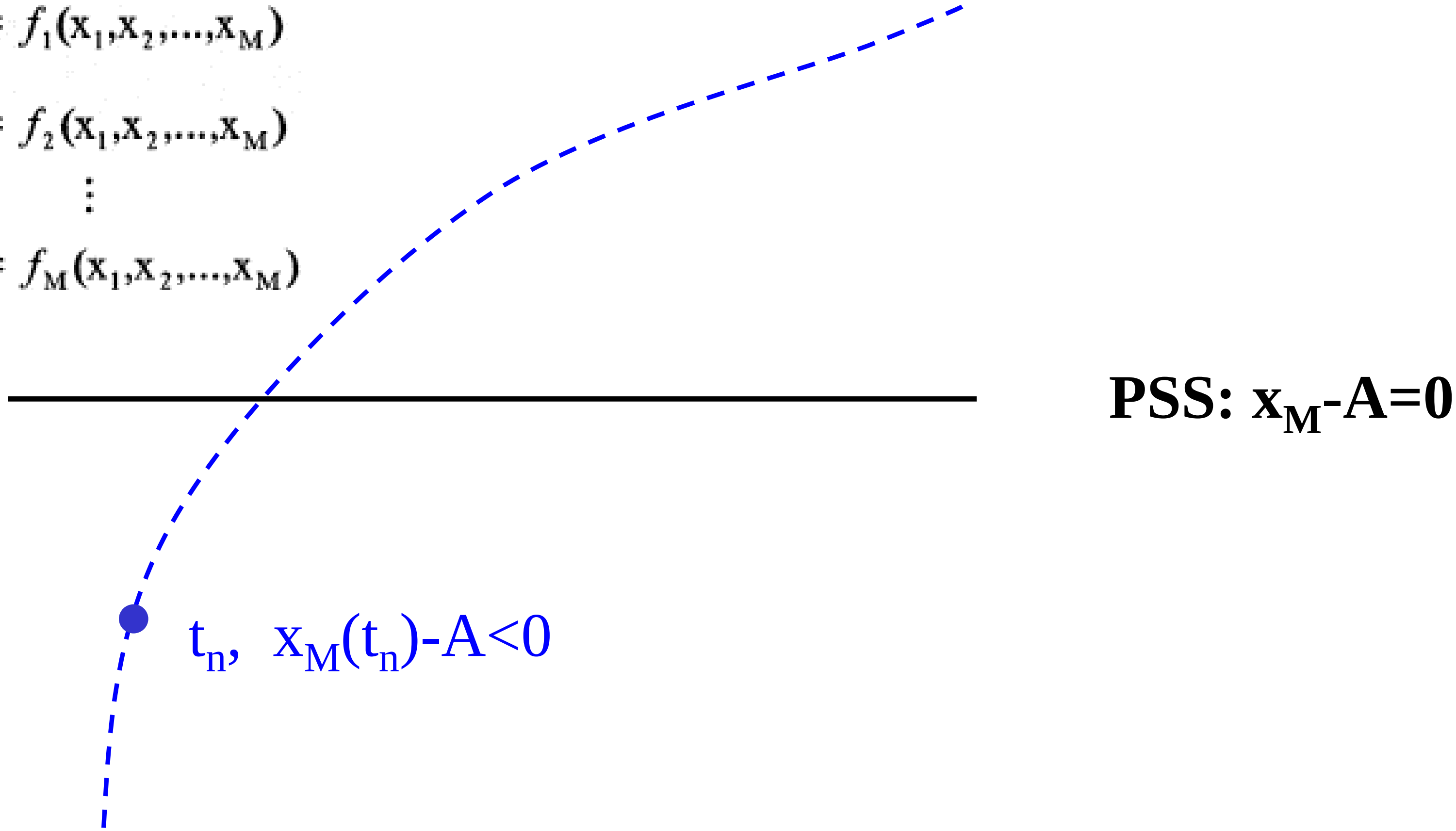

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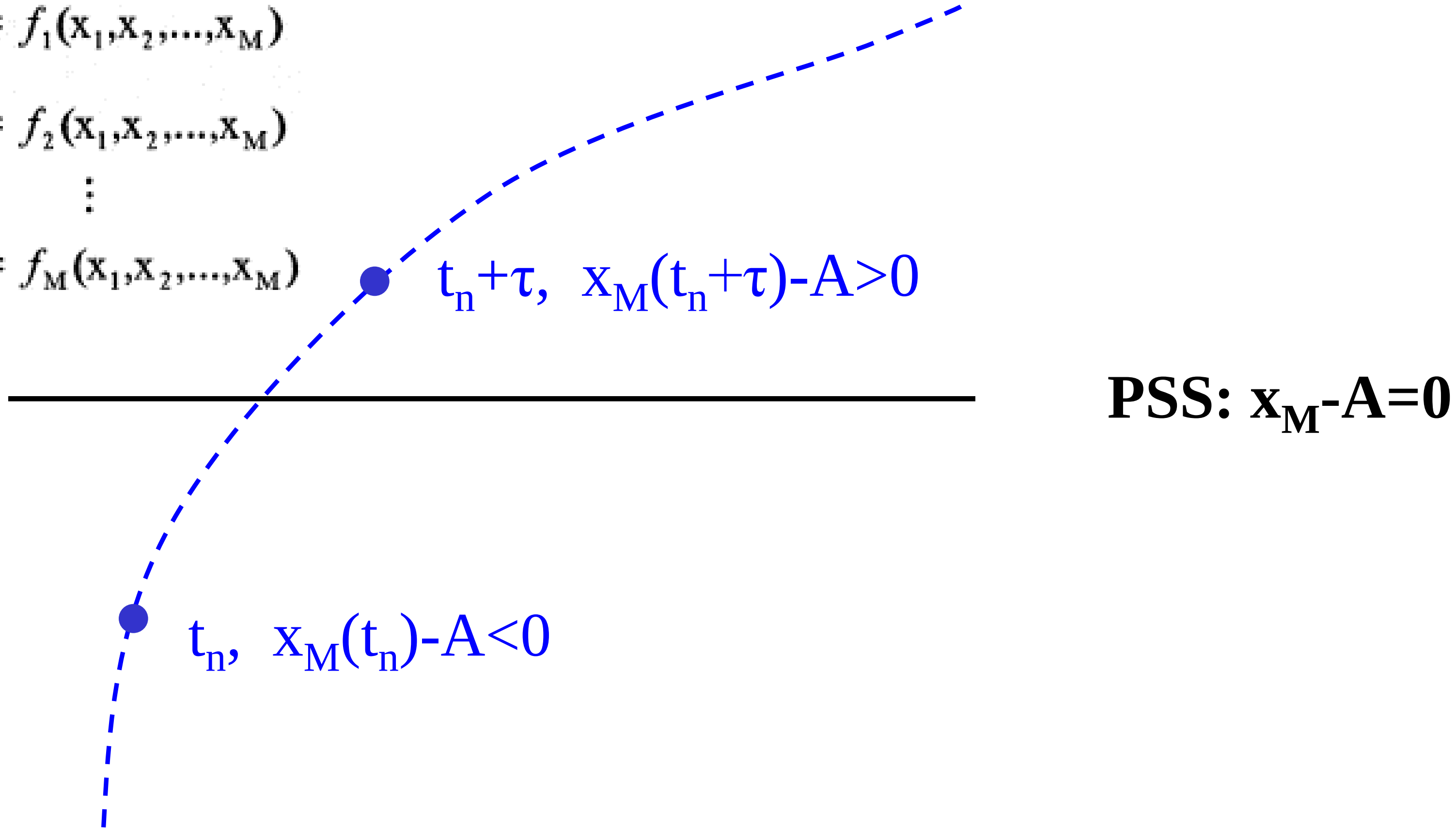

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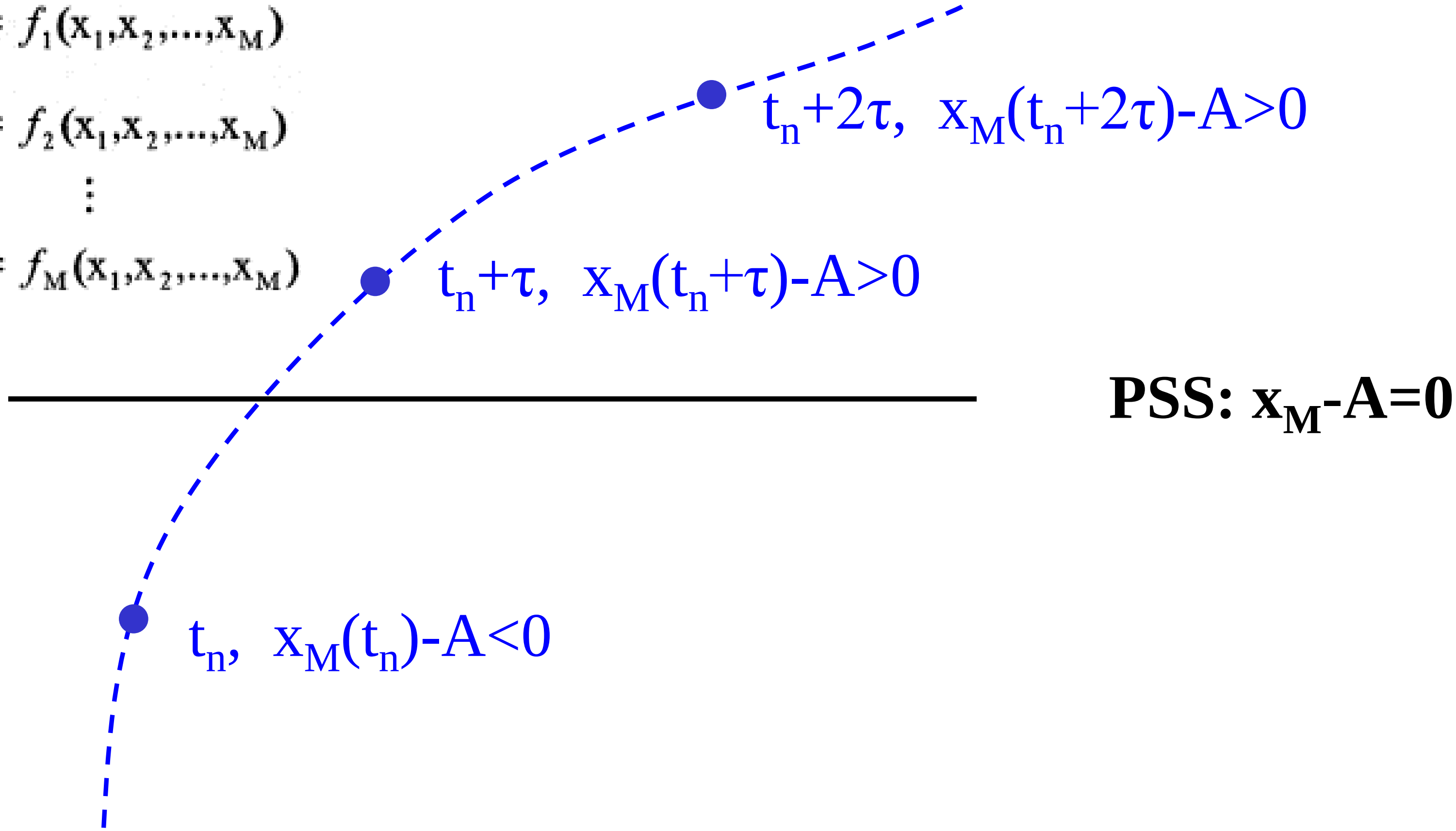

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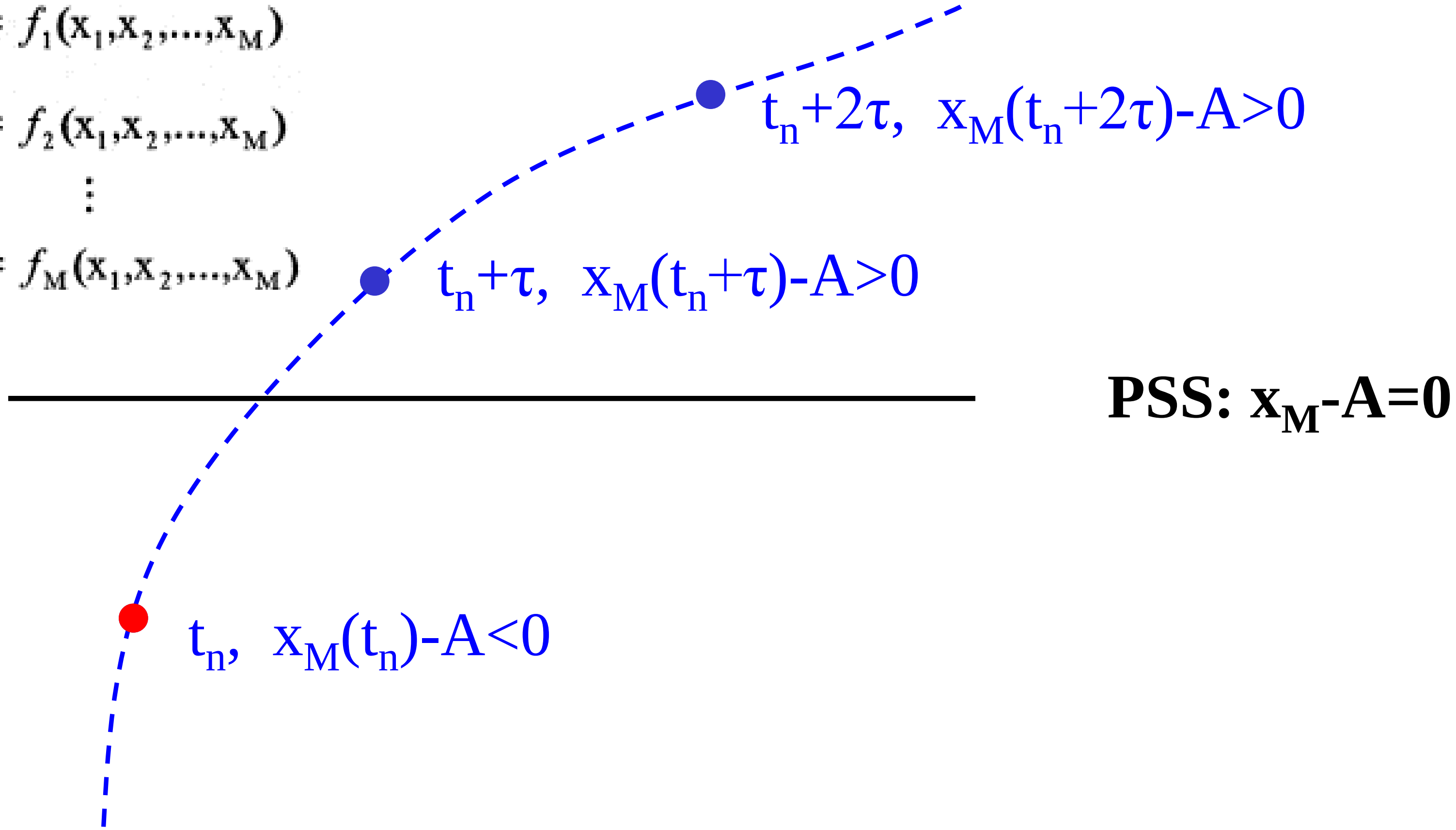

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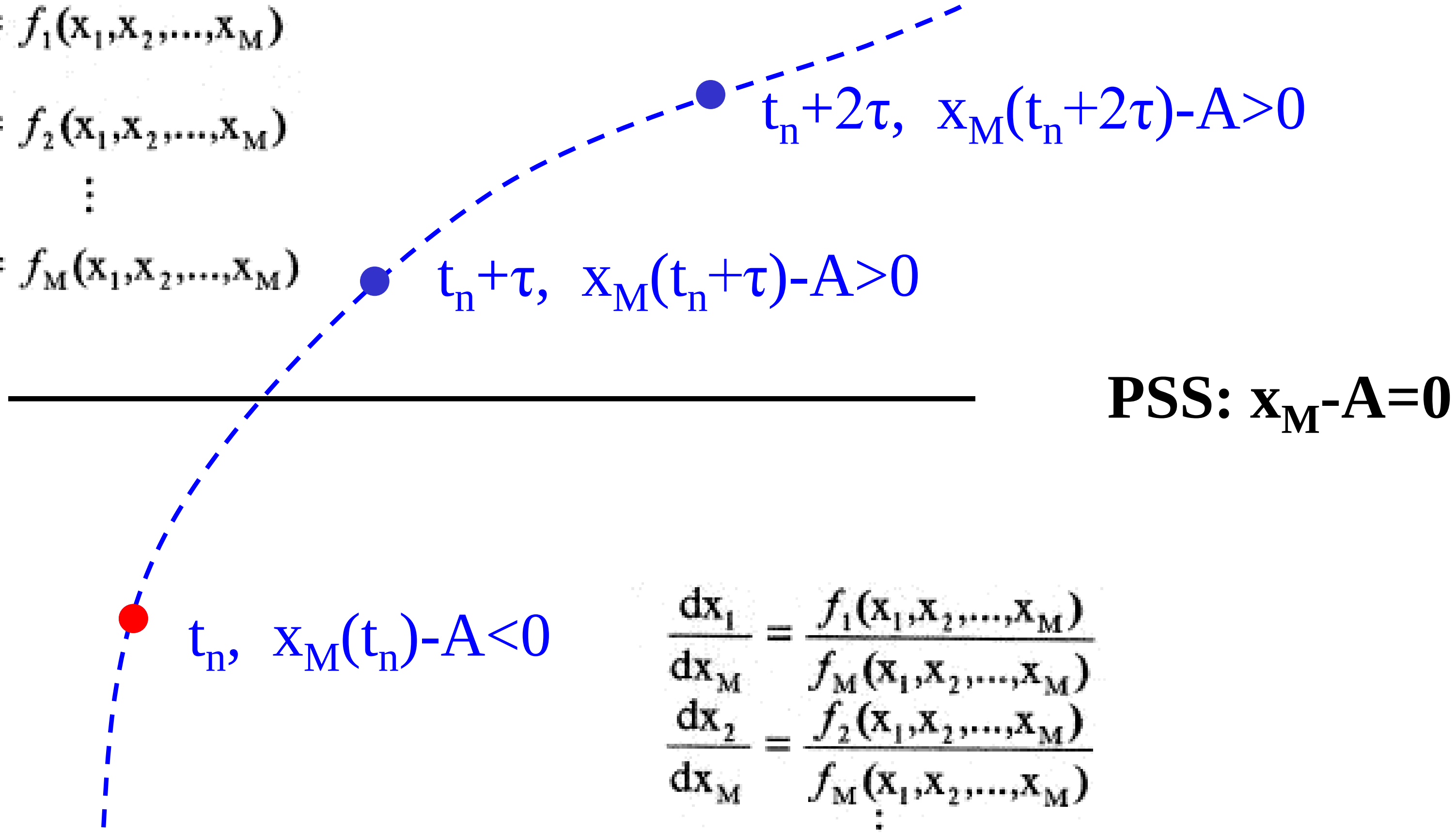

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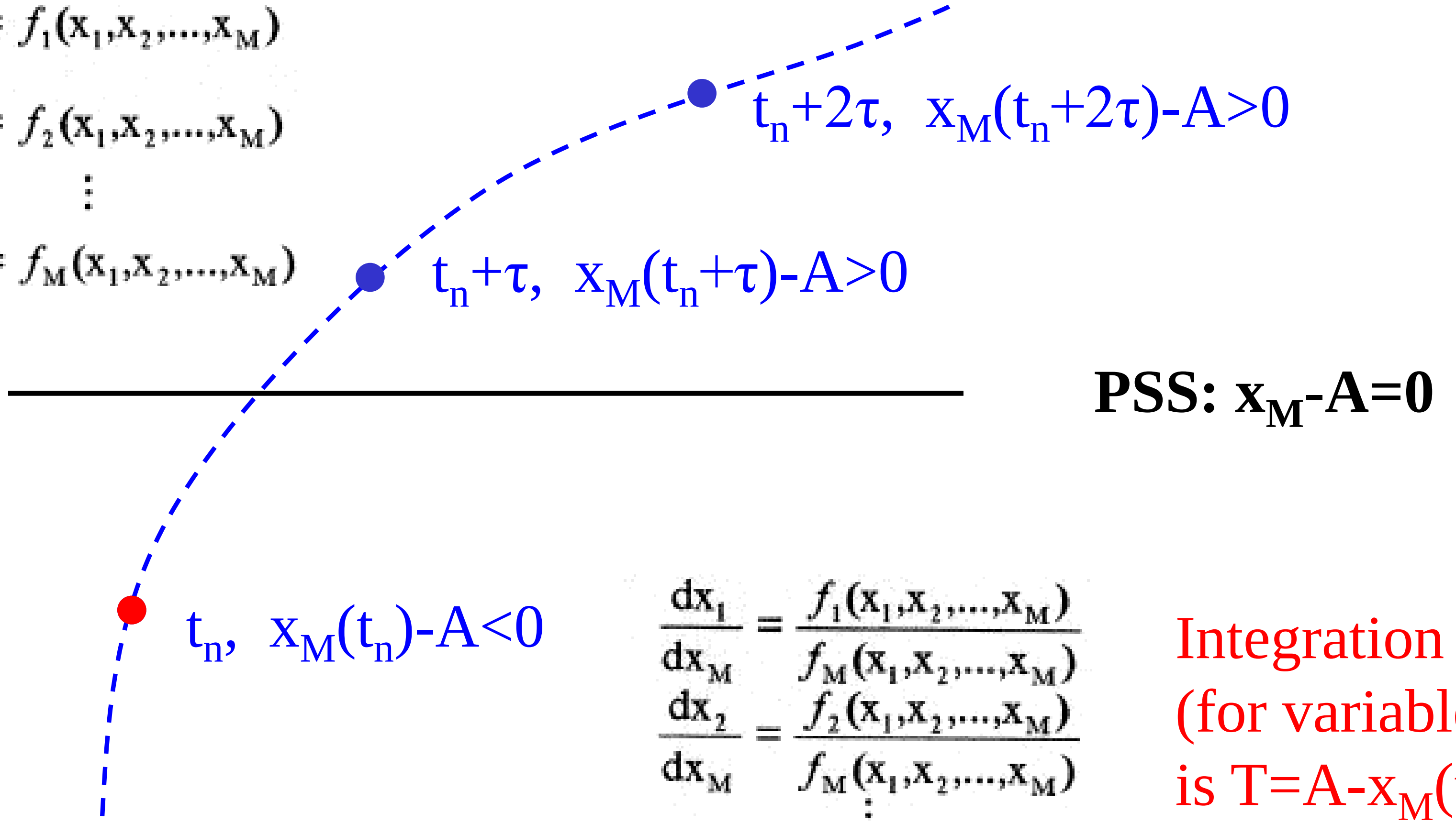

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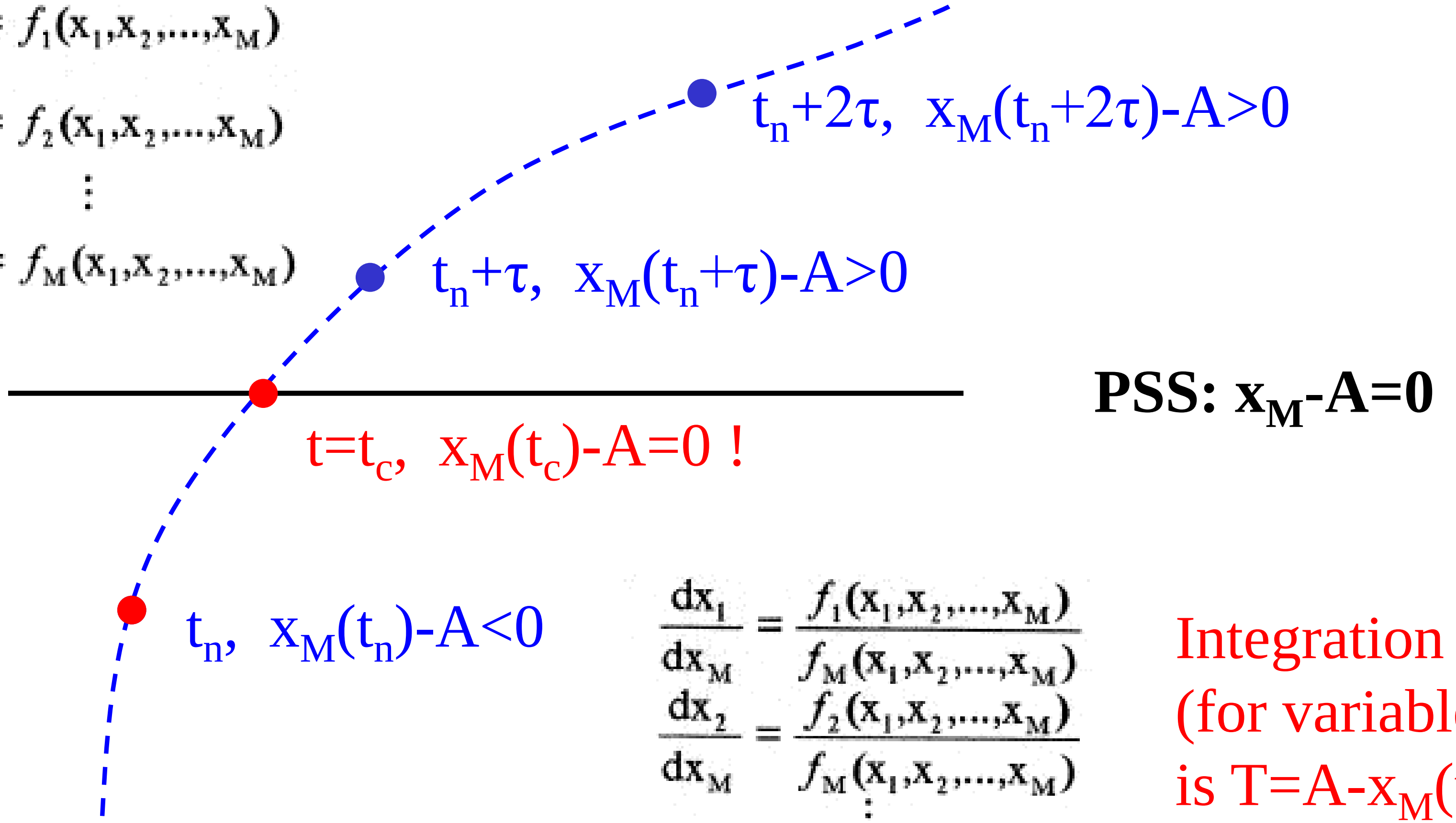

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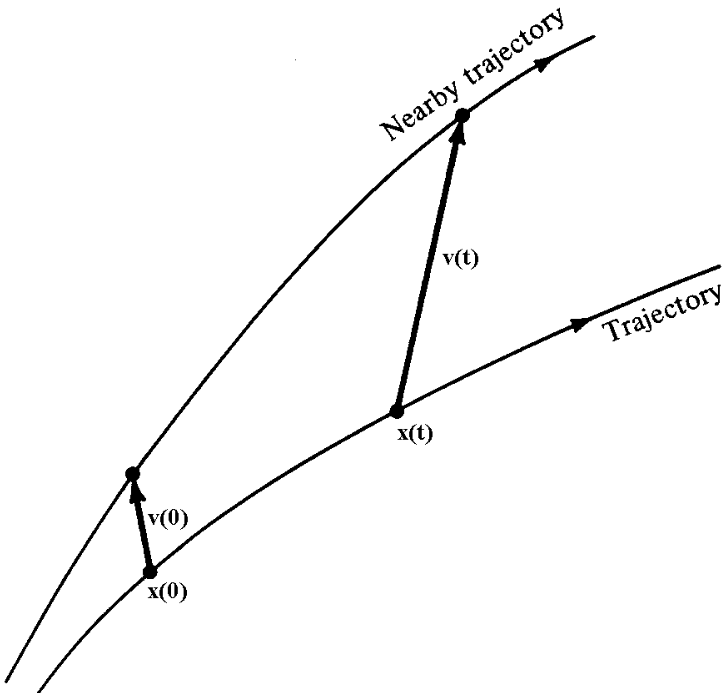
Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (\delta x_1, \delta x_2,\ldots,\delta x_n)^T \text{, with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called **variational equations:**

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}, \quad P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i, j = 1, 2, \ldots, n$$

Example (Hénon-Heiles system)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

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$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2(x + v_1)(y + v_2) \Rightarrow$$

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Variational equations: $\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\dot{v}_1 = v_3$$

$$\dot{v}_2 = v_4$$

$$\boxed{\dot{v}_3 = -v_1 - 2xv_2 - 2yv_1}$$

$$\dot{v}_4 = -v_2 - 2xv_1 + 2yv_2$$

Example (Hénon-Heiles system)

Variational equations: $\frac{dv}{dt} = -J \cdot P \cdot v$

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\begin{aligned} \dot{v}_1 &= v_3 \\ \dot{v}_2 &= v_4 \\ \dot{v}_3 &= -v_1 - 2xv_2 - 2yv_1 \\ \dot{v}_4 &= -v_2 - 2xv_1 + 2yv_2 \end{aligned} \quad + \quad \begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= -y - x^2 + y^2 \end{aligned}$$

Complete set of equations

Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

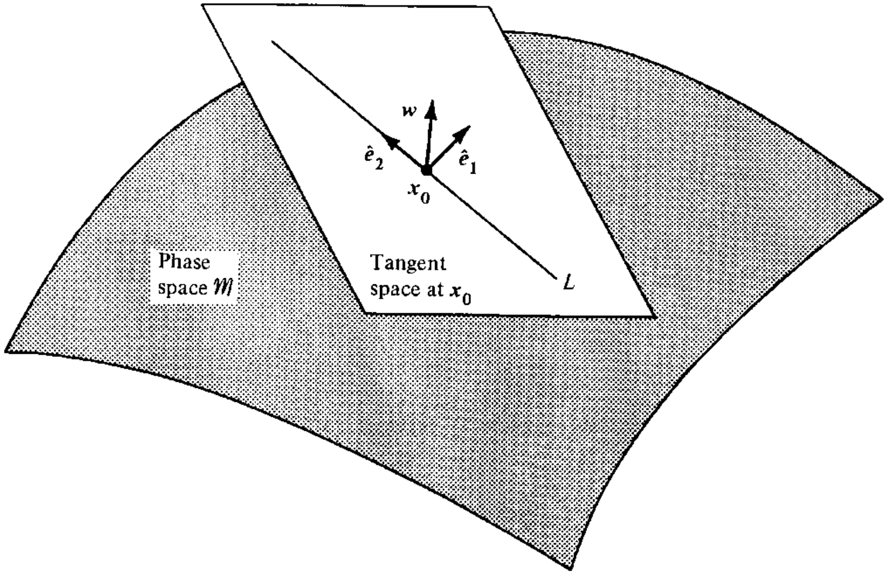
We commonly use the Euclidian norm and set d(0)=||v(0)||=1

Lyapunov Exponents

There exists an **M-dimensional basis {$\hat{e}_i$}** of v such that for any v, σ takes one of the M (possibly nondistinct) values

$$\sigma_i(x(0)) = \sigma(x(0), \hat{e}_i)$$

which are the **Lyapunov exponents.**



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

In autonomous Hamiltonian systems the M exponents are ordered in pairs of opposite sign numbers and two of them are 0.

Computation of the Maximum Lyapunov Exponent

Due to the exponential growth of v(t) (and of d(t)=||v(t)||) we renormalize v(t) from time to time.

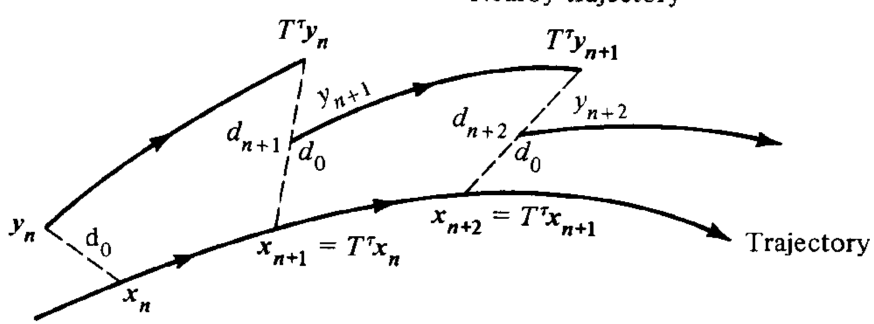


Figure 5.6. Numerical calculation of the maximal Liapunov characteristic exponent. Here $y = x + v$ and τ is a finite interval of time (after Benettin *et al.*, 1976).

Then the Maximum Lyapunov exponent is computed as

$$\sigma_1 = \lim_{n \to \infty} \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i$$

Maximum Lyapunov Exponent

$\sigma_1 = 0 \rightarrow$ **Regular motion**

$\sigma_1 \neq 0 \rightarrow$ **Chaotic motion**

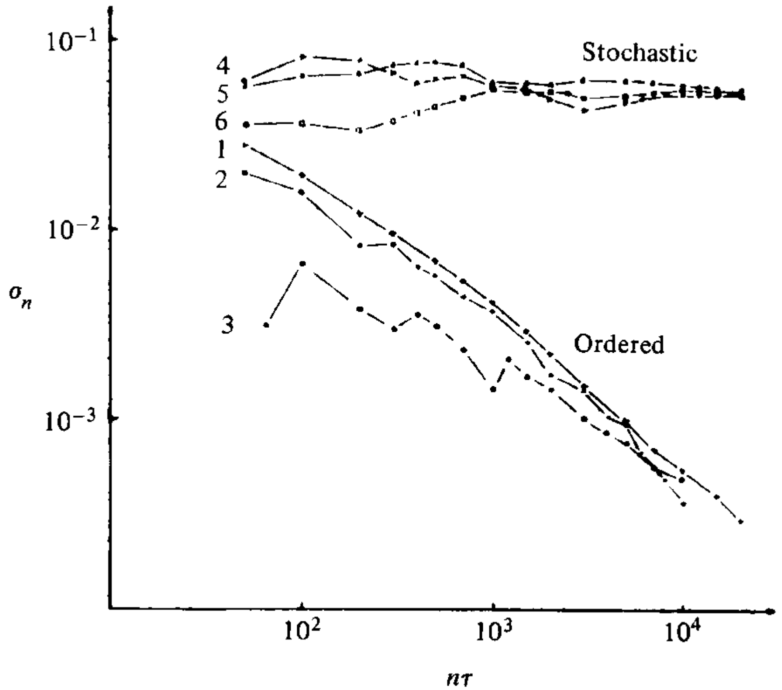
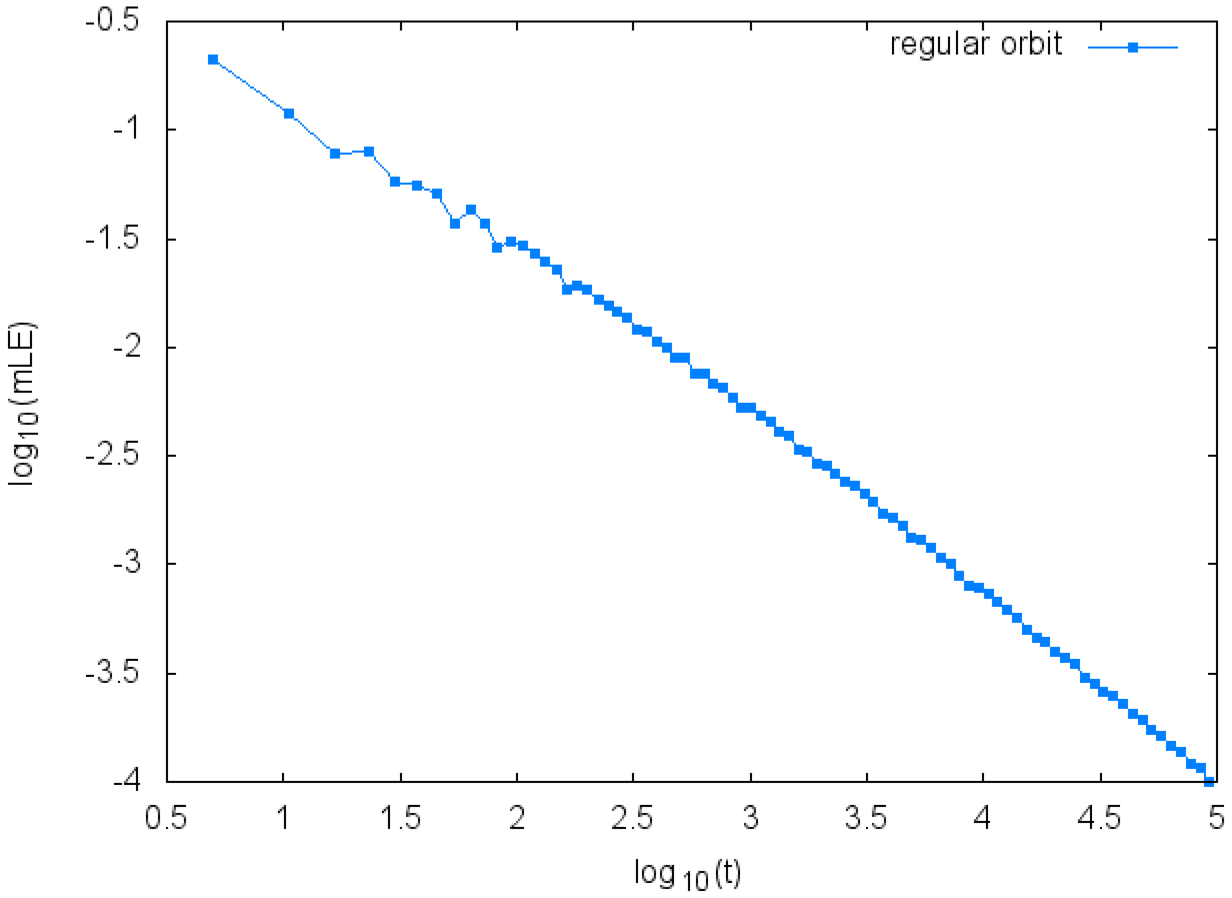


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al*., 1976).

If we start with more than one linearly independent deviation vectors they will align to the direction defined by the largest Lyapunov exponent for chaotic orbits.

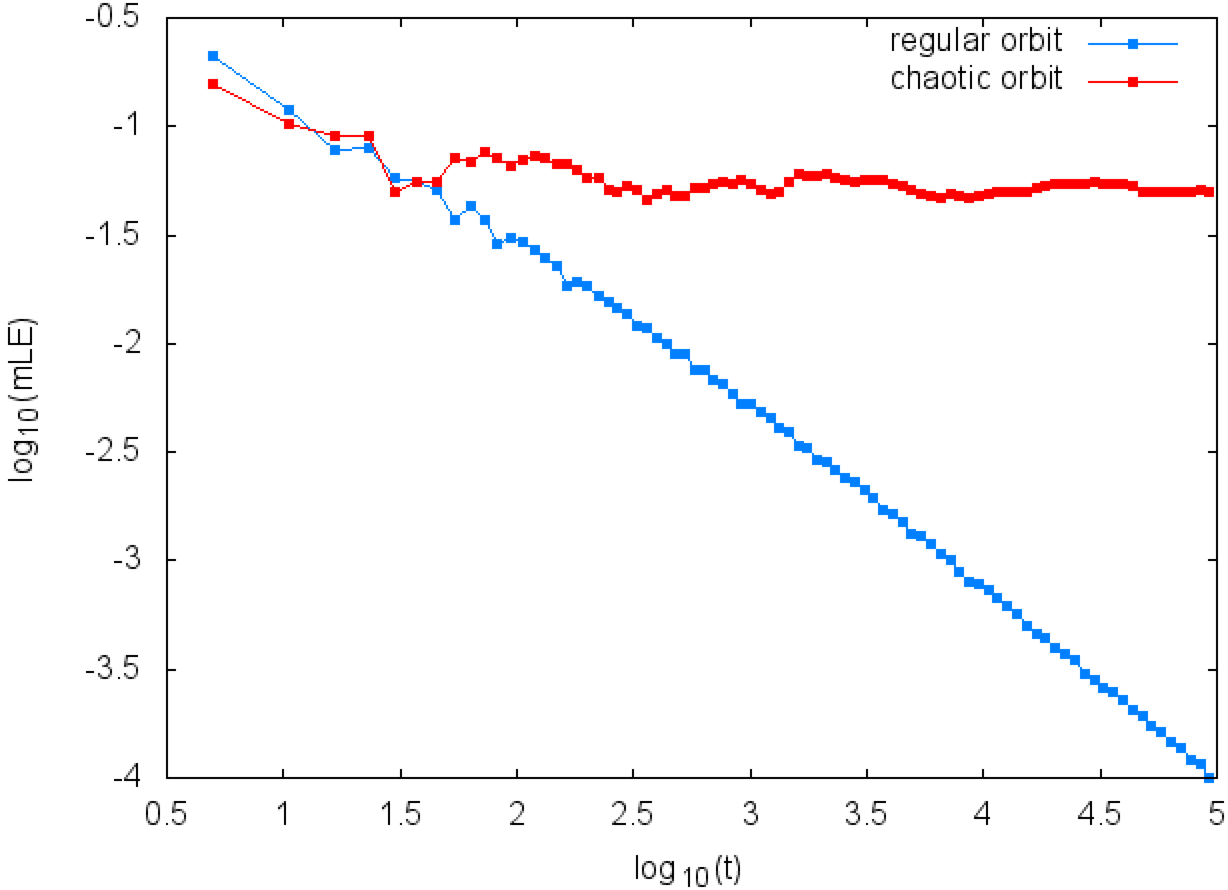
Maximum Lyapunov Exponent

Hénon-Heiles system: Regular orbit



Maximum Lyapunov Exponent

Hénon-Heiles system: Regular orbit and Chaotic orbit



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